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value of $x^2/2y$, derived from $f(x, y) = 0$, and evaluated for $x = 0$, $y = 0$, will be the radius of curvature at the origin; or if the curve is similarly tangent to the y-axis at the origin, $y^2/2x$, evaluated for $x = 0$, $y = 0$, is the radius of curvature at the origin.

SOLUTION BY A. M. HARDING, University of Arkansas.

The equation of any curve having simple tangency to the axis of x at the origin may be written in the form

$$f(x, y) = Ay + \tfrac{1}{2}Bx^2 + Cxy + \tfrac{1}{2}Dy^2 + \cdots = 0,$$

where $A, B, C, D, \cdots$ are constants and $A \neq 0$, $B \neq 0$.

The radius of curvature at any point is given by

$$r^2 = \frac{[f_x^2 + f_y^2]^3}{[f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2]^2}, \qquad \text{where} \qquad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, \text{ etc.}$$

When $x = 0$ and $y = 0$, we find

$$f_x = 0, \qquad f_y = A, \qquad f_{xx} = B, \qquad f_{xy} = C, \qquad f_{yy} = D.$$

Substituting, we obtain $r^2 = A^2/B^2$ or $r = A/B$. Dividing the given equation by B, we have

$$\frac{A}{B} + \frac{x^2}{2y} + Cx + \frac{Dy}{2B} + \cdots = 0.$$

Letting x and y approach zero,

$$\frac{A}{B} = -\left[\frac{x^2}{2y}\right]_{\substack{x=0\\y=0}}.$$

Hence,

$$r = -\left[\frac{x^2}{2y}\right]_{\substack{x=0\\y=0}}.$$

In the second case the method is the same and the equation has the form

$$f(x, y) = Ax + \tfrac{1}{2}Bx^2 + Cxy + \tfrac{1}{2}Dy^2 + \cdots = 0.$$

Also solved by HORACE OLSON in a special form and with incorrect interpretation of "simple tangency."

389. Proposed by FRANK R. MORRIS, Glendale, Calif.

A man is at the southeast corner of a section of land and wishes to walk to the opposite corner in the least possible time. A circular track with a radius of $1/\pi$ miles is located in the section tangent to the west line at a point 120 rods from the south line. Conditions are such that he can walk at the rate of 4 miles an hour inside the track and 3 miles an hour outside the track. What course should he choose and how long is it?

I. SOLUTION BY H. S. UHLER, Yale University.

Since the problem involves rectilinear motion at different speeds it is a question of refraction and can be solved at once by the methods of geometrical optics; for, by Fermat's principle, the time taken by light in going from the southeast to the northwest corner of the section will be either a minimum or a maximum (in this case, a minimum). The index of refraction of the medium outside the circle relative to the medium inside the circle is 4/3 (water and air, say). Hence, the "optical invariant" is

$$3 \sin i = 4 \sin r. \tag{1}$$

By hypothesis, (Fig. 1).

$\overline{OW} = \overline{WN} = 1$ mile. $\overline{CA} = \overline{CB} = \overline{CT} = 1/\pi$ miles, and $\overline{PC} = 120$ rods $= \frac{3}{8}$ mile.

The $\triangle ACB$ is isosceles so that $\angle CBA = \angle CAB \equiv i$. Hence, by equation (1) $\angle EBN = \angle DAO \equiv r$.

The projection of the broken line $OPCA$ on a diameter $\overline{FG}$ perpendicular to $\overline{OA}$ equals zero.

Hence,

$$\overline{IH} + \overline{HC} + \overline{CI} = 0; \qquad \overline{IH} = \overline{JP} = \overline{OP}\sin\phi = (1 - 1/\pi)\sin\phi;$$

$$\overline{HC} = -\overline{PC}\cos\phi = -\tfrac{3}{8}\cos\phi; \qquad \overline{CI} = -\overline{AC}\sin r = -1/\pi\sin r.$$

Therefore,

$$8(\pi - 1)\sin\phi - 3\pi\cos\phi - 8\sin r = 0. \tag{2}$$

In like manner, the projection of the broken line $NTCB$ on a diameter $\overline{KL}$ perpendicular to $\overline{NB}$ leads to

$$8\sin(\phi + 2i - 2r) - 5\pi\cos(\phi + 2i - 2r) + 8\sin r = 0. \tag{3}$$

Equations (1), (2), and (3) determine the acute angles ϕ, i, and r uniquely. Elimination of i and r would lead to a rational equation of degree 16 in $\tan\phi$. The coefficients would involve inconveniently high powers of π and the required root would have to be found by some method of *approximation*, such as Horner's. Consequently no mathematical self-respect is lost and an enormous amount of time is saved by employing the following process of approximation.

Assume a numerical value of ϕ and calculate r from equation (2). Next evaluate i from (1). Then substitute ϕ, i, and r in (3). Repeat until the left member of (3) vanishes. It will then be found that

$$\phi = 37^\circ\ 33'\ 10''; \qquad i = 29^\circ\ 40'\ 27''; \qquad r = 21^\circ\ 47'\ 45''.$$

Let the rectangular coördinates of A and B be (x_1, y_1) and (x_2, y_2) respectively, where abscissas and ordinates are reckoned positive westward and northward in the order named.

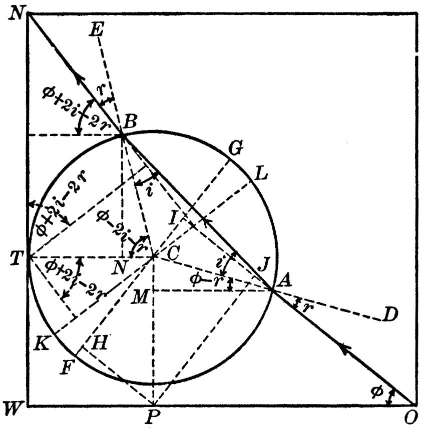

FIG. 1.

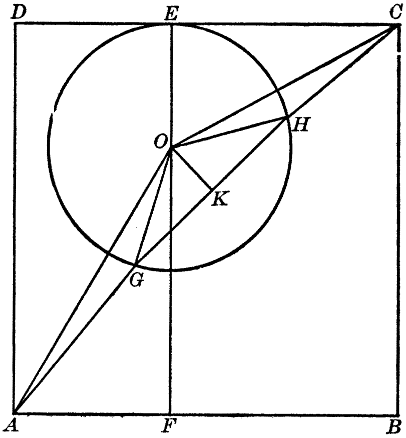

FIG. 2.

$$x_1 = \overline{OP} - \overline{AM} = \left(1 - \frac{1}{\pi}\right) - \frac{1}{\pi}\cos(\phi - r) = 0.375342 \text{ mile},$$

$$y_1 = x_1\tan\phi = 0.288561 \text{ mile}.$$

$$l_1 \equiv \overline{OA} = x_1\sec\phi = 0.473444 \text{ mile}, \qquad l_2 \equiv \overline{AB} = \frac{2}{\pi}\cos i = 0.553130 \text{ mile},$$

$$x_2 = \overline{OW} - (\overline{CT} - \overline{CN}) = 1 - \frac{1}{\pi} + \frac{1}{\pi}\cos(\phi + 2i - r) = 0.763511 \text{ mile}.$$

$$y_2 = \overline{PC} + \overline{NB} = \frac{3}{8} + \frac{1}{\pi}\sin(\phi + 2i - r) = 0.682615 \text{ mile},$$

$$l_3 \equiv \overline{BN} = \overline{NT}\sec(\phi + 2i - 2r) = 0.395803 \text{ mile}.$$

Hence,

$$l_1 + l_2 + l_3 = 1 \text{ mi. } 135 \text{ rd. } 0 \text{ yd. } 2 \text{ ft. } 8 \text{ in.},$$

$$t_1 = 9 \text{ min. } 28.13 \text{ sec.}, \qquad t_2 = 8 \text{ min. } 17.82 \text{ sec.}, \qquad t_3 = 7 \text{ min. } 54.96 \text{ sec.}$$

Hence,

$$t_1 + t_2 + t_3 = 25 \text{ min. } 40.9 \text{ sec.}$$

Finally, the numerical data satisfy the following check equations, obtained by projection of the path on the coördinate axes,

$$l_1 \cos\phi + l_2 \cos(\phi + i - r) + l_3 \cos(\phi + 2i - 2r) = 1,$$

$$l_1 \sin\phi + l_2 \sin(\phi + i - r) + l_3 \sin(\phi + 2i - 2r) = 1.$$

II. SOLUTION BY A. H. HOLMES, Brunswick, Maine.

Let *ABCD* (Fig. 2) be the section of land, 1 mile square, *A* being the southeast corner, *B* the northeast corner, *C* the northwest corner, and *D* the southwest corner. On *DC*, take $DE = 120$ rods or $\frac{3}{8}$ of a mile and draw *EF* perpendicular to *AB* at *F*. On *EF*, take $EO = 1/\pi$ of a mile to point *O*.

With *O* as a center and *EO* as radius describe a circle. Draw *AO* and *CO*. Let *AG* be the course to the circumference of track, *GH* the course inside the track, and *HC* the remainder of the course to the northwest corner *C*.

Draw *OK* perpendicular to *GH* at *K*. The ratio of *GH* to $AG + CH$ is the greatest when *OK* bisects the angle *AOC*. Therefore, for a minimum, the angles *AOG* and *COH* are equal.

Put $AG = x$, $CH = y$, $GH = z$, $AO = a$, $CO = b$, $1/\pi = r$, $AOG = COH = \theta$, and $AOC = 2m$.

Then

$$\frac{x + y}{3} + \frac{z}{4} = \text{minimum}.$$

In $\triangle AGO$ and *CHO*

$$\cos\theta = \frac{a^2 + r^2 - x^2}{2ar}, \qquad \cos\theta = \frac{b^2 + r^2 - y^2}{2br}.$$

Hence $x = \sqrt{a^2 + r^2 - 2ar\cos\theta}$, $y = \sqrt{b^2 + r^2 - 2br\cos\theta}$, and $z = 2r\sin(m - \theta)$ in which $a = .778027$, $b = .701390$, $r = .318310$, and $m = 72° 54' 9''$.

Multiplying *x* and *y* by 4 and *z* by 3, reducing and differentiating,

$$\frac{2a\sin\theta}{\sqrt{a^2 + r^2 - 2ar\cos\theta}} + \frac{2b\sin\theta}{\sqrt{b^2 + r^2 - 2br\cos\theta}} = 3\cos(m - \theta).$$

Whence, $\theta = 12° 33' 9''$.

Knowing θ, the required course in distances and directions is easily found.

From *A* to *G*, N. 52° 45′ 48″ W. 151.17 rods; *G* to *H*, N. 44° 5′ 13″ W. 177.03 rods; *H* to *C*, N. 37° 1′ 51″ W. 126.96 rods, the length of the whole course being 455.16 rods, and made in the minimum time of 25 minutes 40.8 seconds.

MECHANICS.

303. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

A pile driver weighing 500 pounds falls through 10 feet and drives a pile weighing 400 pounds 3 inches into the ground. Show that the average force of the blow is 11,111$\frac{1}{9}$ pounds.

SOLUTION BY W. H. WILLIAMS, Oakland, California.

Let m = mass of driver in pounds, m' = mass of pile in pounds, v = velocity of driver at instant of impact in feet per second, h = height of fall of driver in feet, d = distance pile is driven into ground in feet, r = average retardation of pile and driver after impact in feet per second per second, t = time from impact until rest in seconds, and g = acceleration due to gravity in feet per second per second, F = average force of blow.

Then mv = momentum of driver at impact = momentum of driver + pile immediately after impact; $mv/(m + m')$ = common velocity of driver and pile immediately after impact.

Hence,

$$\frac{mv}{(m + m')r} = t, \qquad \text{and} \qquad d = \tfrac{1}{2}rt^2 = \tfrac{1}{2}r\frac{m^2v^2}{(m + m')^2r^2} = \tfrac{1}{2}\frac{m^2v^2}{(m + m')^2r},$$